Numerical Integration

Integration on Action:

The velocity of free falling bungee jumper as a function of time can be computed as:

$v(t) = \sqrt{\frac{gm}{c_d}}\tanh(\sqrt{\frac{gc_d}{m}}t).$

Suppose that we would like to know the vertical distance z the jumper has fallen after a certain time t. The distance can be evaluated by integration:

$$z(t) = \int_0^t v(t)dt$$

What is Integration?

- In dictionary integrate means "to indicate the total amount, to unite".
- Mathematically definite integration is represented by: $\int_a^b f(x)dx$, which stands for the integral of the function $f(x)$ with respect to independent variable x, evaluated between the limits $x = a$ to $x = b$.

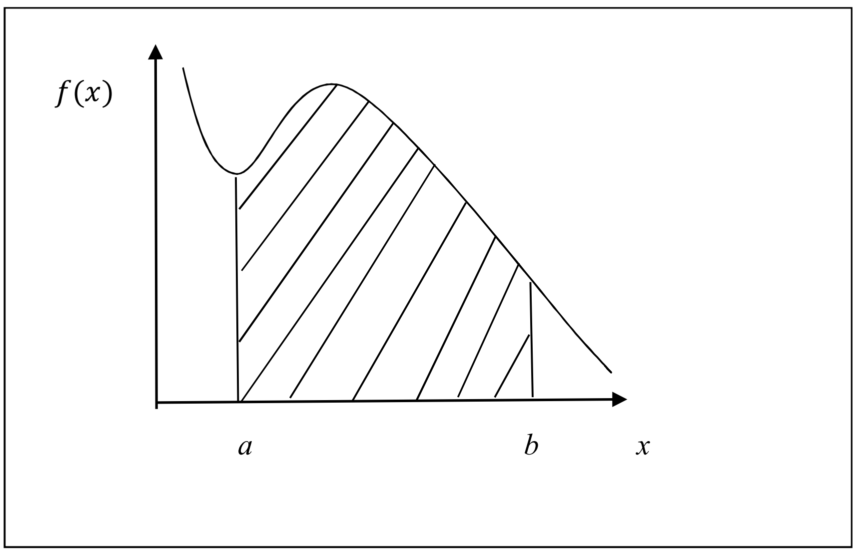

Figure 1: Graphical representation of the integral $f(x)$ between the limits $x = a$ to $x = b$. The integral is equivalent to the area under the curve

Numerical Integration:

A given set of data points (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), ….. ,(x_n, y_n) of a function $y = f(x)$, where *f(x)* is not known explicitly and *f(x)* is determined using the interpolation technique. So, *Numerical integration* is the process of finding the numerical value of a definite integral

$$I = \int_a^b f(x)\ dx$$

when a function $y = f(x)$ is not known explicitly. But we are given only a set of values of the function $y = f(x)$ corresponding to the same values of x.

Newton-Cotes formulas:

The Newton-Cotes formulas are the most common numerical integration schemes. Given a function $f(x)$, defined on an interval [a,b], we want to find an approximate to the integral:

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

Where, $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

Where, n is the order of the polynomial.

Main Idea:

- Cut up [a, b] into smaller sub intervals
- In each sub-interval, find polynomial $p_i(x) \approx f(x)$;
- Integrate $p_i(x)$ on each sub interval and sum them up (degree of polynomial we choose according to the problem);
- For lower degree polynomial error will be more than higher degree polynomial but algorithm will be complicated.

Graphical Example for approximation of different order integrals:

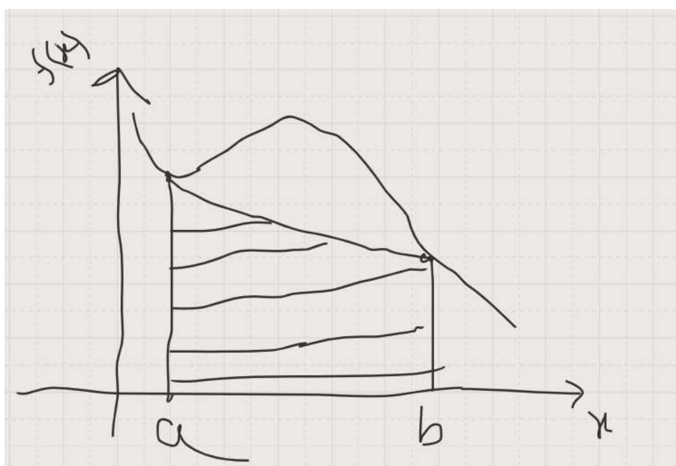

Figure 2: The approximation of integral by the area under single straight line

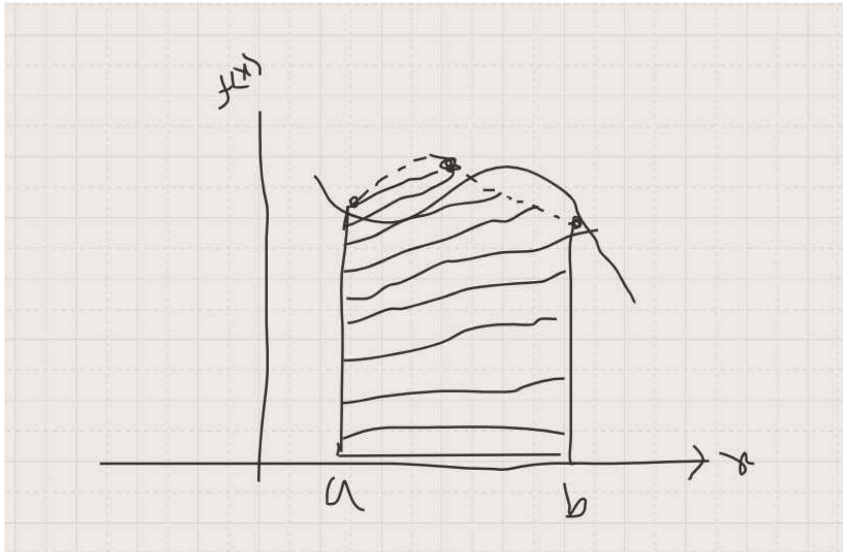

Figure 3: The approximation of integral by the area under single parabola

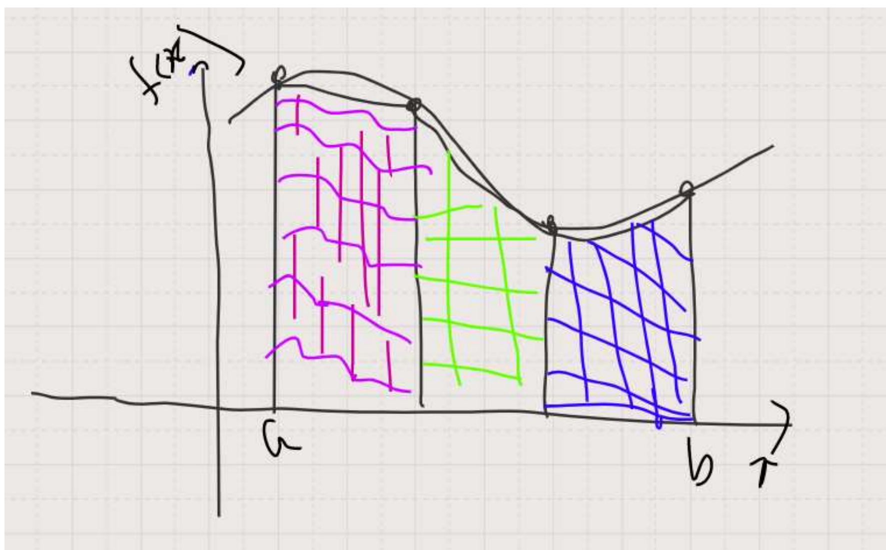

Figure 4: The approximation of integral by the area under by the three straight line segment

Need and Scope:

Like in numerical differentiation, we need to evaluate the numerical integration in the following cases:

- Functions do not possess closed form solutions.
  Example: $f(x) = C\int_0^x e^{-t^2}\,dt$
- Closed form solutions exist but these solutions are complex and difficult to use for calculations.
- Table-shaped data for variables are available, but no mathematical relationship is established between them, as is often the case with experimental data.

- In order to evaluate the integral, we attach an appropriate polynomial of interpolation to the set of values of f(x) and then integrate it within the desired limits. Instead of $f(x)$, we add an approximate formula for interpolation. By applying this technique to a single variable element, the process is called Quadrature.
- If f(x) is continuous over the closed interval [a, b], then the integral I = $\int_a^b f(x)\,dx$ represents the area under the curve y = f(x) bounded by the ordinates x = a, x = b and the x-axis.

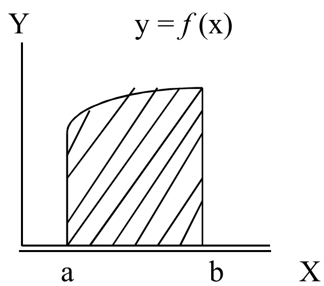

Figure 5: Area under the curve y = f(x)

General Quadrature formula for equidistant ordinates

Suppose y = f(x) is a function whose values are known at equidistant values of x i.e. $f(x_0) = y_0, f(x_0+h) = y_1, f(x_0+2h) = y_2, \ldots\ldots., f(x_0+nh) = y_n$. We have to evaluate,
I = $\int_a^b f(x)\,dx$.
To evaluate I, we have to replace f(x) by a suitable interpolation formula. Let the interval [a, b] be divided into n subintervals with the division points $a = x_0 < x_0+h < \ldots\ldots < x_0+nh = b$ where h is the width of each subinterval.

Approximating f(x) by *Newton's forward interpolation formula* we can write the integral as

$$I = \int_{x0}^{x0+nh} f(x)\,dx$$
$$= \int_{x0}^{x0+nh} (y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{2!}\Delta^2 y_0 + \frac{u\,(u-1)\,(u-2)}{3!}\Delta^3 y_0 + \ldots..)dx \ldots\ldots.(1)$$

Now,
$u = (x - x_0) / h$
$\Rightarrow x = x_0 + u\,h$

$\frac{dx}{du} = h$
$\Rightarrow dx = du h$

At $x = x_0$; $u = 0$ and at $x = x_0+n\,h$; $u = n$
Hence,

$$I = h \int_0^n (y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\,\Delta^3 y_0)du$$

$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2}\Delta^2 y_0 + \frac{\frac{u^4}{4} - 3\frac{u^3}{3} + 2\frac{u^2}{2}}{6}\Delta^3 y_0\right]_0^n$$

$$= h[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2!}\Delta^2 y_0 + \frac{\frac{n^4}{4} - 3\frac{n^3}{3} + 2\frac{n^2}{2}}{3!}\Delta^3 y_0] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

We can obtain different numerical integration formula by assigning n =1, 2, 3 etc in equation 1and n is the number of sub intervals.

Closed and Open form of Newton Cotes Methods:
In-closed form integration limits of integration points are known. On the other hand in open form integration the sets of integration points are beyond the range of data.

Closed form Integration formulas:

1. Trapezoidal Rule
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule
4. Boole's Rule

Trapezoidal Rule
The Trapezoidal rule is the first of the Newton Cotes closed integration formulas and correspond to the case where polynomial is first order.

We cut up interval [a,b] into n sub-intervals $x_0 = a$, $x_i < x_{i+1}$, $x_n = b$. On $[x_i, x_{i+1}]$, approximate $f(x)$ by a linear polynomial p_i.

so, $p_i(x) = f(x_i)$ and $p_{i+1}(x) = f(x_{i+1})$

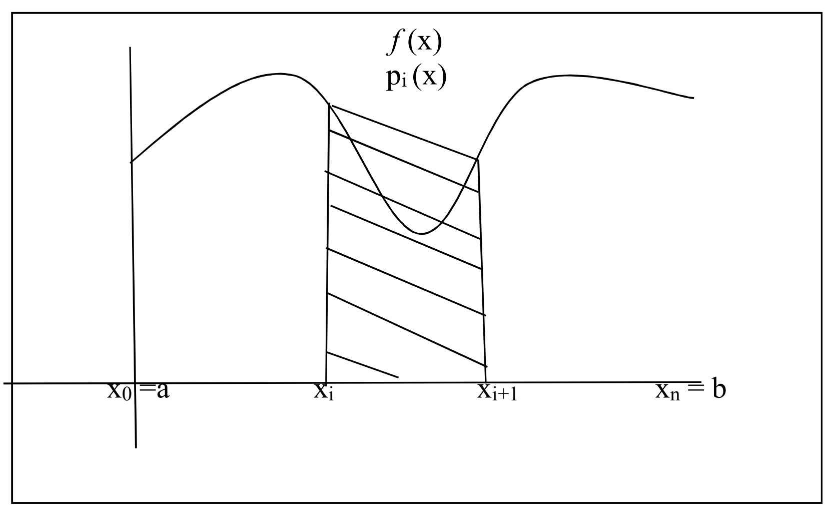

Figure 5: Graphical interpretation of Trapezoidal Rule

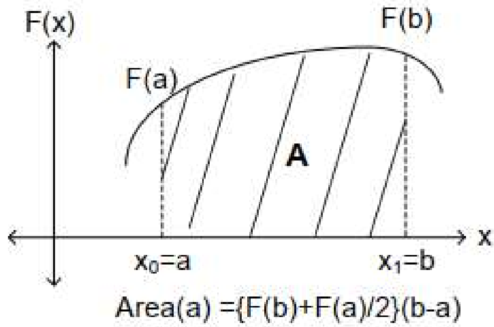

Figure 6: Area of the Trapezoidal

The straight line passing through the points ($f(a)$, a) and ($f(b)$, b) can be represented as:

$$\frac{f(x)-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a}$$

$$\therefore f(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$

The area under this straight line is an estimate of the integral of f(x) between the limits a and b:

$$\mathrm{I}=\int_a^b [f\,(\mathrm{a})\;+\;\frac{f(b)-f(a)}{b-a}\;(x-a)]dx$$

The result of the integration is:

$$I=\;(b-a)\,\frac{f(b)+f(a)}{2}$$, which is called trapezoidal rule.

$$I = \text{width} * \text{average height}$$

$$I\;=\;\text{width} * \frac{f(b)+f(a)}{2}$$

Derivation of Trapezoidal Rule:

$$I=\int_{x0}^{x0+nh}\;(y_0+u\,\Delta y_0+\frac{u\,(u-1)}{2!}\Delta^2\,y_0+\frac{u\,(u-1)\,(u-2)}{3!}\Delta^3\,y_0+\ldots..)dx\;\ldots\ldots.(1)$$

Trapezoidal rule is two points formula, it 1st order polynomial $p_1(x)$ for approximating the function $f(x)$. From equation (1) the integral for trapezoidal rule is given by taking 1st two terms:

$$I=\int_a^b(y_0+\;u\,\Delta y_0)dx=\int_a^b(y_0)dx+\int_a^b u\,\Delta y_0\,dx\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

Let,

$dx = h * du$

$x_0 = a;\; x_1 = b \text{ and } h = b - a$

$At\ x = a, u = \dfrac{(a - x_0)}{h} = 0$

$At\ x = b, u = \dfrac{(b - x_0)}{h} = 1$

$$\int_a^b (y_0)\mathrm{dx} = \int_0^1 h(y_0)\mathrm{du} = \mathrm{h}y_0$$
$$\int_a^b \mathrm{u}\,\Delta y_0\,\mathrm{dx} = \int_0^1 \Delta y_0\,\mathrm{uh\,du} = \mathrm{h}\frac{\Delta y_0}{2}$$

Therefore,

$$\boldsymbol{I} = \mathrm{h}(y_0 + \frac{\Delta y_0}{2}) = h\left(\frac{2y_0 + y_1 - y_0}{2}\right) = h\frac{y_1 + y_0}{2} = h\frac{fx_1 + fx_0}{2} = h\frac{f(b) + f(a)}{2}$$
$$= (b-a)\frac{f(b) + f(a)}{2}$$

Error of the Trapezoidal Rule:
An estimation of local truncation error of trapezoidal rule is:

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where, $\boldsymbol{\xi}$ lies somewhere in the interval from a to b. Equation 4 indicate that if the function is being integrated as linear function than error zero otherwise for higher order derivatives some error may occur.

Example: Numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from *a* = 0 to *b*=0.8 using Trapezoidal Rule. Note that the exact value of the intergral can be determined analytically to be 1.6405333.

Solution:
Given *a* = 0 and *b*=0.8
Therefore,
$f(a=0) = 0.2 + 25 * 0 - 200(0)^2 + 675(0)^3 - 900(0)^4 + 400(0)^5 = 0.2$
$f(b=0.8) = 0.2 + 25 * 0.8 - 200(0.8)^2 + 675(0.8)^3 - 900(0.8)^4 + 400(0.8)^5 = 0.232$
According to trapezoidal rule:

$$I = (b-a)\frac{f(b) + f(a)}{2} == (0.8-0)\frac{(0.232) + (0.2)}{2} = 0.1728$$

Relative % Error: $\frac{1.6405333 - 0.1728}{1.6405333} * 100 = 89.46\%$
The reason for large error is depicted in Figure 7. Notice that the area under the straight line do not cover significant portion of the integral lying above the line.

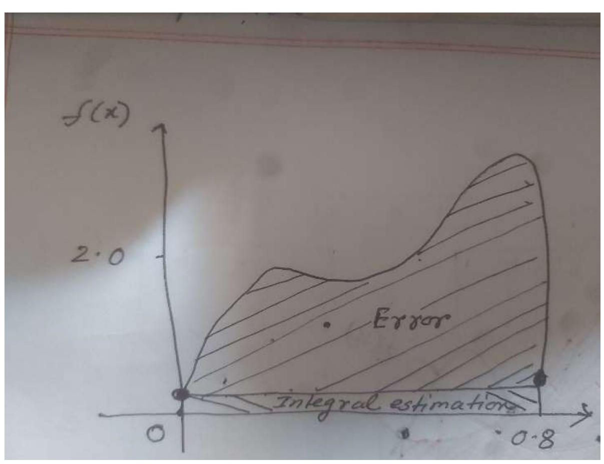

Figure 7: Graphical depiction of single application of trapezoidal rule to approximate the intergral.

Derivation of Composite Trapezoidal Rule from Newton Cotes formula:

One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into number of segment and apply method for each segment. The area of each intervals are added to get integration for entire interval. The resulting equation is known as composite integration formula.

Setting, n = 1 in General Quadrature Formula (2) and neglecting second and higher order differences gives a two point formula by which numerical integration is carried out known as Trapezoidal Rule.

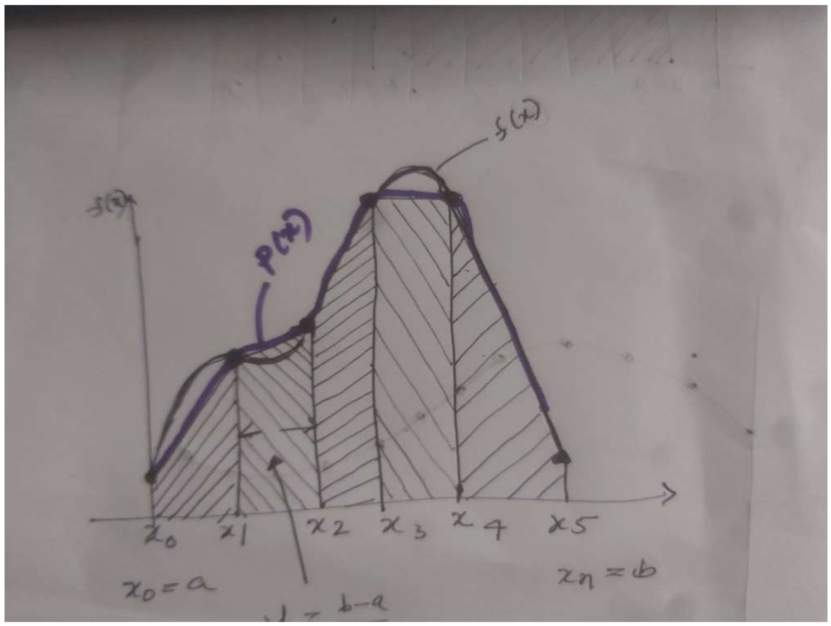

Figure 8: Illustration of Composite Trapezoidal Rule

Substituting n =1 in the general quadrature formula (1) and neglecting all differences greater than the first, we get

$$I_1 = \int_{x0}^{x0+h} f(x)\,dx$$

$$= h\,[y_0 + 1/2\ \Delta y_0] = h\,[y_0 + 1/2\ (y_1 - y_0)] = \frac{h}{2}(2y_0 + y_1 - y_0)$$

$$= \frac{h}{2}(y_0 + y_1)$$

for the first sub interval $[x_0, x_0+h]$

Similarly, we get

$$I_2 = \int_{x0+h}^{x0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$I_3 = \int_{x0+2h}^{x0+3h} f(x)\,dx = \frac{h}{2}(y_2 + y_3).$$

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$$I_n = \int_{x0+(n-1)h}^{x0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Combining all these expressions, we obtain

$$I = I_1 + I_2 + \ldots\ldots\ldots\ldots + I_n$$
$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots\ldots\ldots\ldots + \frac{h}{2}(y_{n-1} + y_n)$$
$$= \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots\ldots\ldots + y_{n-1}) + y_n]$$

The above formula is known as the composite *trapezoidal rule* for numerical integration.`

- The geometrical significance (figure 8) of this rule is that the curve $y = f(x)$ is replaced by n straight line joining the points (x_0, y_0) and (x_1, y_1), (x_1, y_1) and (x_2, y_2), …. , (x_{n-1}, y_{n-1}) and (x_n, y_n).
- The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$ and the x axis is then approximately equivalent to the sum of the areas of the n trapezium obtained.

Example: Calculate the value $\int_0^1 \frac{x}{(1+x)} dx$ correct up to 3 significant figures taking six intervals by trapezoidal rule.

Solution: Here we have $f(x) = \frac{x}{(1+x)}$ $a = 0$; $b = 1$ $n = 6$

$$\therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

X	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = f(x)$	0.0	0.14286	0.25000	0.33333	0.40000	0.45454	0.50000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The trapezoidal rule can be written as

$$I = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$
$$= \frac{1}{12}[(0.0 + 0.5) + 2(0.14286 + 0.25 + 0.33333 + 0.4 + 0.45454)]$$
$$= 0.30512$$

$\therefore$ I = 0.305, correct to 3 significant figures.

Example: Numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to b=0.8 using Composite Trapezoidal Rule. Note that the exact value of the integral can be determined analytically to be 1.6405333.

Solution:
By taking 2 segment of intervals i.e. n = 2

$$h = \frac{b-a}{n} = \frac{0.8-0}{2} = 0.4$$

Therefore,

$$f(a = 0) = 0.2;\ f(0.4) = 2.456;\ f(b = 0.8) = 0.232$$

$$I = \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{h}{2}[0.2 + 2(2.456) + 0.232] = 1.0688$$

Relative % Error: $\frac{1.6405333 - .0688}{1.6405333} * 100 = 34.9\%$

Simpson's 1/3 (One-third) Rule

Simpson's 1/3 Rule is for more accurate estimation of an integral by using higher order polynomials to connect the points. For example if there is an extra m point midway between$f$ (a) and $f(b)$, the three points can be connected with a parabola (figure 9) and if there are two points between f (a) and $f(b)$, the four points can be connected with a parabola (figure 10). The formulas that result from taking the integral under these polynomials are called Simpson's rule.

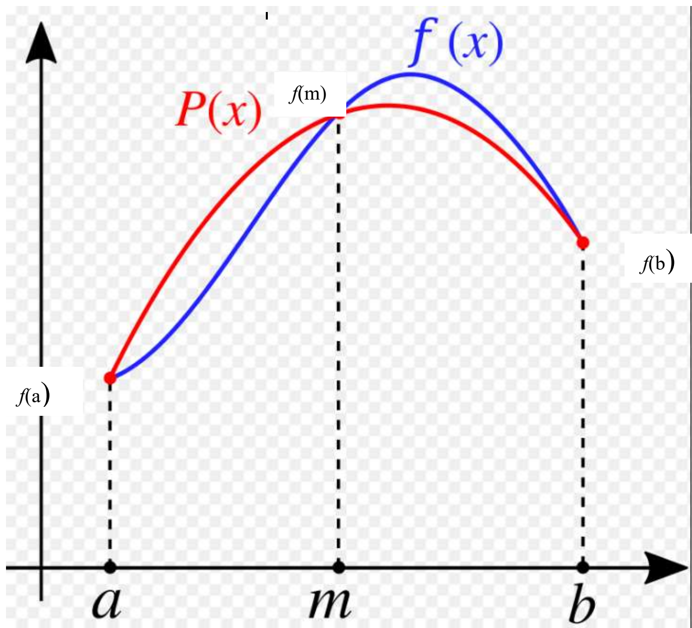

Figure 9: Illustration of Simpson's 1/3 rule

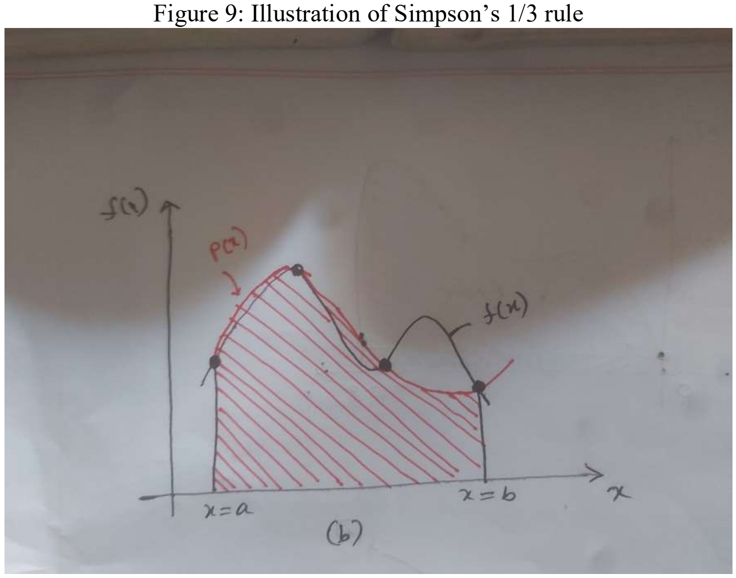

Figure 10: Illustration of Simpson's 3/8 rule

Simpson's 1/3 rule results when a second order interpolating polynomial into equation:

$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by second order Lagrange polynomial, the integral becomes:

$$I = \int_a^b [\frac{(x-x1)(x-x2)}{(x0-x1)(x0-x2)}f(x0) + \frac{(x-x0)(x-x2)}{(x1-x0)(x1-x2)}f(x1) + \frac{(x-x0)(x-x1)}{(x2-x0)(x2-x1)}f(x2)]dx$$

After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}[f(x0) + 4f(x1) + f(x2)]$$

Where, for this case $h = \frac{(b-a)}{2}$, $x_1 = \frac{a+b}{2}$ and the equation known as Simpson's 1/3 rule.

Derivation of Composite Simpson's 1/3 (One-third) Rule

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3}-\frac{n^2}{2}}{2!}\Delta^2 y_0 + \frac{\frac{n^4}{4}-3\frac{n^3}{3}+2\frac{n^2}{2}}{3!}\Delta^3 y_0\right] \ldots\ldots\ldots\ldots\ldots(1)$$

Here, the function f(x) is approximated by a second order polynomial, which passes through three sampling points. So, taking n = 2 in general quadrature formula and neglecting third and higher order differences, we get:

$$I_1 = \int_{x0}^{x0+2h} f(x)\,dx$$

$$= h[n\,y0 + \frac{n^2}{2}\Delta y0 + \frac{\frac{n^3}{3}-\frac{n^2}{2}}{2!}\Delta^2 y_0]$$

$$= h[2\,y_0 + 2^2/2\,\Delta y_0 + (2^3/3 - 2^2/2)\,\Delta^2 y_0/2!]$$

$$= h[2\,y_0 + 2^2/2\,\Delta y_0 + (8/3 - 2)\,\Delta^2 y_0/2]$$

$$= h\,[2y_0 + 2\Delta y_0 + 1/3\,\Delta^2 y_0]$$

$$= h\,[2y_0 + 2\,(y_1 - y_0) + 1/3\,(y_2 - 2y_1 + y_0)]$$

$$= 1/3\,h\,[6y_0 + 6y_1 - 6y_0 + y_2 - 2y_1 + y_0]$$

$$= h/3\,(y_0 + 4y_1 + y_2)$$

Since, it uses three points to evaluate the integral; it is also known as three-point formula. The three points are: x_0 = a, x_2 = b and x_1 = (a + b)/2. Where, the interval 'h' is given by: h = (b – a)/2.

Similarly, we get

$$I_2 = \int_{x0+2h}^{x0+4h} f(x)\,dx = h/3\,(y_2 + 4y_3 + y_4)$$

$$I_3 = \int_{x0+4h}^{x0+6h} f(x)\,dx = h/3\,(y_4 + 4y_5 + y_6)$$

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$$I_{n/2} = \int_{x0+(n-2)h}^{x0+nh} f(x)\,dx = h/3\,(y_{n-2} + 4y_{n-1} + y_n)$$

n intervals $h = \frac{(b-a)}{n}$

Combining all these expressions, we obtain

$$I = I_1 + I_2 + \ldots\ldots\ldots\ldots + I_n$$
$$= h/3\,[(y_0+4y_1+y_2) + (y_2+4y_3+y_4) + \ldots\ldots\ldots + (y_{n-2}+4y_{n-1}+y_n)]$$
$$= h/3\,[y_0 + 4\,(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2\,(y_2 + y_4 + \ldots + y_{n-2}) + y_n]$$

The above formula is known as *Simpson's one-third rule* on simply *Simpson's rule*.

- It should be noted that this rule requires the division the whole range into an *even number of subintervals* of width h.

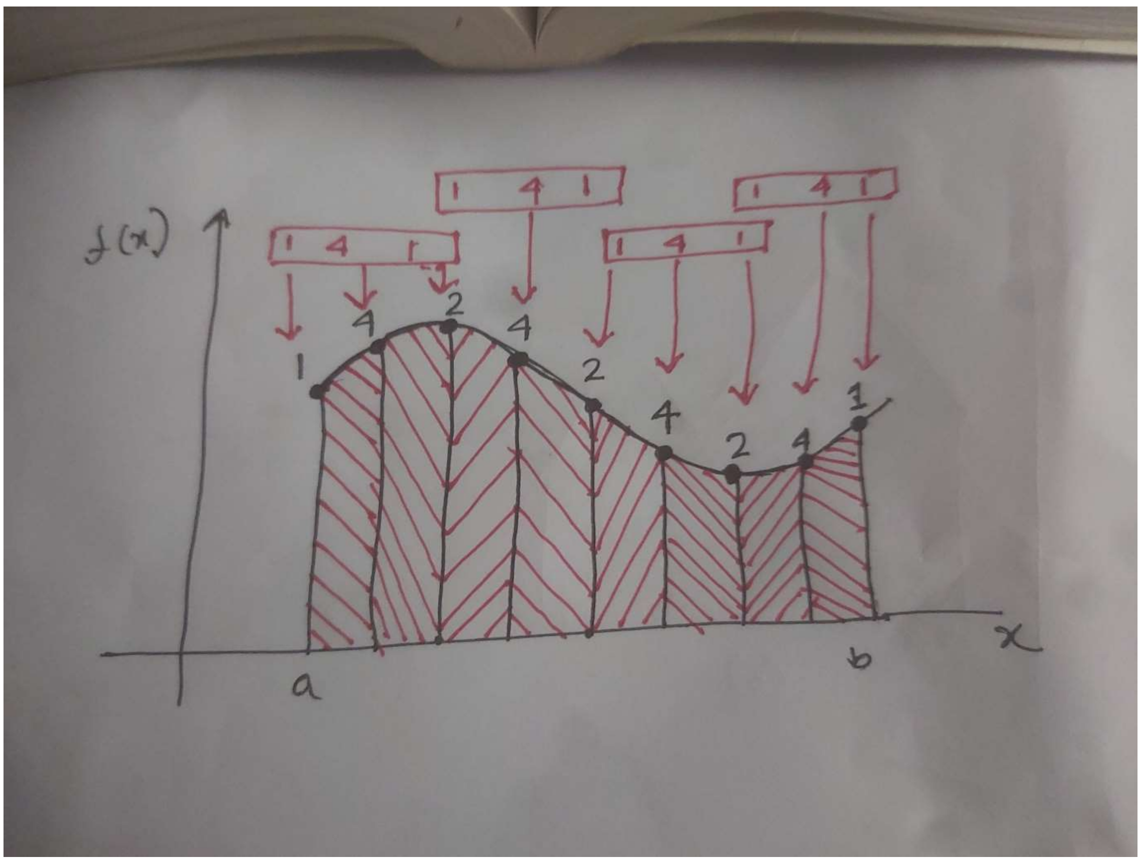

Figure 11: Illustration of Composite Simpson's 1/3 Rule

Notice that, as illustrated at figure 11 that the odd points represent the middle term for each application and hence carry weight of 4. The even points are common to adjacent applications and hence carry weight 2.

Example: Estimate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ with n=4 from a = 0 to b = 0.8. Recall that the exact integral is 1.640533.

Solution:
n=4, h = (0.8 − 0)/4 = 0.2

$f(0)$	$f(0.2)$	$f(0.4)$	$f(0.6)$	$f(0.8)$
0.2	1.288	2.456	3.464	0.232

$$I = \frac{0.2}{3}(0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232)$$

$$= 1.623467$$

Estimated error is $E_t = \frac{1.640533 - 1.62345}{1.640533} * 100 = 1.04\ \%$

Simpson's 3/8 (Three-eight) Rule

Simpson's 1/3 rule was derived using three sampling points that fit a quadratic equation. We can extent this approach to incorporate four sampling points so that the rule can be exact for $f(x)$ of degree 3.

Substituting n = 3 and $h = \frac{b-a}{3}$ in the general quadrature formula and neglecting all the differences above Δ^3, we get

$$I_1 = \int_{x0}^{x0+3h} f(x)\,dx$$
$$= h\,[3y_0 + 9/2\ \Delta y_0 + (9 - 9/2)\ \Delta^2 y_0 / 2 + (81/4 - 27 + 9)\ \Delta^3 y_0 / 6]$$
$$= 3h\,[y_0 + 3/2\ \Delta y_0 + 3/4\ \Delta^2 y_0 + 1/8\ \Delta^3 y_0]$$
$$= 3h\,[y_0 + 3/2\ (y_1 - y_0) + 3/4\ (y_2 - 2y_1 + y_0) + 1/8\ (y_3 - 3y_2 + 3y_1 - y_0)]$$
$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, we get

$$I_2 = \int_{x0+3h}^{x0+6h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$
$$I_3 = \int_{x0+6h}^{x0+9h} f(x)\,dx = \frac{3h}{8}(y_6 + 3y_7 + 3y_8 + y_9)$$

.

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$$I_{n/3} = \int_{x0+(n-3)h}^{x0+nh} f(x)\,dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Combining all these expressions, and $h = \frac{b-a}{n}$ we obtain

$$I = I_1 + I_2 + \ldots\ldots\ldots\ldots + I_n$$
$$= \frac{3h}{8}[(y_0+3y_1+3y_2+y_3) + (y_3+3y_4+3y_5+y_6) + \ldots\ldots\ldots\ldots + (y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)]$$

$= \frac{3h}{8} [(y_0 + y_n) + 3 (y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-2} + y_{n-1}) + 2 (y_3 + y_6 + \ldots + y_{n-3})]$

- Simpson's 3/8 rule can be applied when the range [a, b] is divided into a number of subintervals, which must be a *multiple of 3*.

Example:
Estimate the integral of $f(x) = x^3 + 1$ with n=3 from a = 1 to b = 2 using Simpson's 3/8 rule.

Solution:
n=3, $h = \frac{2-1}{3} = \frac{1}{3}$
So, four points are
$x_0 = 1 \ \therefore y_0 = (1)^3 + 1 = 2$
$x_1 = (1 + 0.33) = 1.33 \ \therefore y_1 = (1.33)^3 + 1 = 3.352637$
$x_2 = (1.33 + 0.33) = 1.66 \ \therefore y_2 = (1.66)^3 + 1 = 5.574296$
$x_3 = 2 \ \therefore y_3 = (2)^3 + 1 = 9$

$f(1)= y_0$	$f(1.33)= y_1$	$f(1.66)= y_2$	$f(2)= y_3$
2	3.352637	5.574296	9

$$I = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3]$$

$$I = \frac{3}{8*3}(2 + 3*(3.352637 + 5.574296) + 9)$$

$= 4.7225$

Weddle's rule:

Substituting n = 6 in the general quadrature formula and neglecting all differences above Δ^6, we get
$I = 3h/10 [(y_0+y_n) + (y_2+y_4+y_8+y_{10}+\ldots\ldots+y_{n-4}+y_{n-2}) + 5 (y_1+y_5+y_7+y_{11}+\ldots\ldots +y_{n-5}+y_{n-4})$
$+ 6 (y_3+y_9+y_{15}+ \ldots\ldots +y_{n-3}) + 2 (y_6+y_{12}+ \ldots\ldots\ldots\ldots +y_{n-6})]$

- Substituting n = 4 in the general quadrature formula and neglecting all differences above Δ^4, we get the *Boole's rule*.

Example: Find the value of $\int_4^{5.2} log_e\, x dx$ by Weddle's rule.
Solution:
Here $f(x) = log_e x$ a = 4 b= 5.2 taking n=6 we have

$h = \frac{5.2-4}{6} = 0.2$

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y = f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6457

Weddle's rule is:

$I = \int_4^{5.2} log_e\, xdx$ = 3h/10 [(y_0 + 5 (y_1 + y_5) + y_2 + 6 y_3 + y_6] = 0.06[304643] = 1.827858

Summary:

Basic Newton Cotes Rule:

Name	Intervals (n)	Formula	Error
Trapezoidal Rule	1	$I = \frac{h}{2}(y_0 + y_1)$	$-\frac{h^3}{12}f''(\theta)$
Simpson's 1/3 Rule	2	$I = \frac{h}{3}(y_0 + 4y_1 + y_2)$	$-\frac{h^5}{90}f''''(\theta)$
Simpson's 3/8 Rule	3	$I = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$	$-\frac{h^5}{80}f''''(\theta)$
Boole's Rule	4	$I = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$	$-\frac{8h^7}{945}f''''''(\theta)$

More Examples on Simpson's Rule from Text Book [2]:

Example: The velocity of a train which starts from rest given in the following table, the time is recorded in minutes and velocity in km/hr. Estimate the total distance run in 20 minutes.

t(s)	0	2	4	6	8	10	12	14	16	18	20
v(t)	0	16	28.8	40	46.4	51.2	32.0	17.6	3.2	16	0

Solution:

$$v = \frac{ds}{dt} => ds = v.dt$$
$$=> \int ds = \int v.dt$$
$$s = \int_0^{20} v.dt$$

The train starts from rest,

The velocity $v = 0$ when $t = 0$, here n = 10

$$h = \frac{b-a}{n} = \frac{20-0}{10*60} = \frac{20}{60}\, hrs = \frac{1}{30} hrs.$$

The Simpson's rule is:

$$s = \int_0^{20} v.dt = \frac{h}{3}[(y_0 + y_1) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$
$$= \frac{1}{30 \times 3}[(0+0) + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)]$$
$$= \frac{1}{90}[0 + 4 \times 128 + 2 \times 115.2] = 8.25\, km$$

The distance run by the train 20 minutes is 8.25 km

Example:
Evaluate $\int_0^1 \frac{1}{1+x^2}$, by taking seven ordinates using Simpson's three-eights rule.

Solution: We have,

$$n + 1 = 7 => n = 6$$

The points of division are,

$$0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$$

X	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x^2}$	1.000000	0.9728730	0.900000	0.800000	0.6923077	0.590163	0.50000

$$here\ h = \frac{1}{6}, the\ Simpson's\ three - eighths\ rule\ is$$

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2 * y_3]$$

$$= \frac{3}{6 \times 8}[(1.000000 + 0.500000) + 3(0.9728730 + 0.900000 + 0.6923077 + 0.590163) + 2(0.800000)]$$

$= \frac{1}{16}[1.5000000 + 9.4663338 + 1.6000000]$

= 0.7853959

References:

1. **Steven C. Chapra, Raymont P. Canale.** *Numerical Methods for Engineer.* New Delhi : Tata McGraw Hill Publishing company Ltd, 2003. 4th Edition.

2. **BalaGurushamy, E.** *Numerical Methods.* New Delhi : Tata McGraw-Hill, 2000.