EXAMPLES OF NUMERICAL INTEGRATION

Raqeebir Rab

EXAMPLE 1:

Evaluate the integral $\int_0^2 (3x^3 + 2x^2 - 1)dx$ using:

A. single segment Trapezoidal Rule

B. Basic Simpson's 1/3 Rule

C. Basic Simpson's 3/8 Rule

EXAMPLE 1:

Case A. Trapezoidal Rule: $I = \frac{h}{2}[f(a) + f(b)]$

a=0
b=2
$f(x) = 3x^3 + 2x^2 - 1$

$$I = \frac{h}{2}[f(a) + f(b)]$$
$$= \frac{b-a}{2}[f(0) + f(2)]$$
$$= \frac{2-0}{2}[(-1) + (31)] = 30$$

Error:

$$|E_t| \leq \frac{h^3}{12} max|f''(x)|; 0 \leq x \leq 2$$

$$f''(x) = 18x + 4 = 40$$

$$\therefore |E_t| \leq \frac{h^3}{12} * 40 = \frac{2^3}{12} * 40 = 26.666$$

$$I_{exact} = 15.333$$

relative % error

$$= \left|\frac{15.333 - 30}{15.333}\right| * 100\% = 95.65\%$$

EXAMPLE 1:

Case B. Simpson 1/3 Rule: $I = \frac{h}{3}\,[f(x0) + 4f(x1) + f(x2)]$

a=0= $x0$

b=2= $x2$

$h = \frac{b-a}{2}$ =1

$$x1 = \frac{a+b}{2} = 1$$

$$f(x) = 3x^3 + 2x^2 - 1$$

$$I = \frac{h}{3}\,[f(x0) + 4f(x1) + f(x2)]$$

$$= \frac{1}{3}[f(0) + 4f(1) + f(2)]]$$

$$=\frac{1}{3}[(-1) + 4 * (4) + (31)] = 15.333$$

$$I_{exact} = 15.333$$

relative % error

$$= \left|\frac{15.333 - 15.333}{15.333}\right| * 100\% = 0\%$$

EXAMPLE 1:

Case C. Simpson 3/8 Rule: $I = \frac{3h}{8}\left[f(a) + 3f(x1) + 3f(x2) + f(b)\right]$

a=0

b=2

$$h = \frac{b-a}{3} = \frac{2-0}{3} = \frac{2}{3}$$

$$x1 = 0 + \frac{2}{3} = \frac{2}{3}$$

$$x2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$f(x) = 3x^3 + 2x^2 - 1$$

$$I = \frac{3h}{8}\left[f(a) + 3f(x1) + 3f(x2) + f(b)\right]$$

$$= \frac{3h}{8}\left[f(0) + 3f\left(\frac{2}{3}\right) + 3f\left(\frac{4}{3}\right) + f(2)\right]$$

$$= \frac{3}{8} * \frac{2}{3}\left[(-1) + 3 * \left(\frac{7}{9}\right) + 3 * \left(\frac{119}{9}\right) + (31)\right]$$

$$= \frac{1}{4}\left[(-1) + \left(\frac{126}{3}\right) + (31)\right] = \frac{72}{4} = 18$$

$$I_{exact} = 15.333$$

relative % error

$$= \left|\frac{15.333 - 18}{15.333}\right| * 100\% = 17.4\%$$

EXAMPLE 2:

Estimate the integral $\int_1^3 \frac{dx}{x}$ using for n=4 and n=6:

A. Composite Trapezoidal Rule

B. Composite Simpson's 1/3 Rule

C. Composite Simpson's 3/8 Rule

Exact Value of $\int_1^3 \frac{dx}{x} = 1.09861$

Solution for n=4:

Here,

$a = 1$

$b = 3$

n=4

$h = (b - a)/n = (3-1)/4 = 1/2$

EXAMPLE 2:

A. Composite Trapezoidal Rule: $I = \frac{h}{2}[y_0 + \left(2\sum_{i=1}^{n-1} f_i\right) + f_n]$

x	1	1.5	2	2.5	3
$f(x) = \frac{1}{x}$	1	0.6666	0.5	0.4	0.3333
	y_0	y_1	y_2	y_3	y_4

$$\therefore I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$$

$$= \frac{1}{4}[1 + 2(0.6666 + 0.5 + 0.4) + 0.3333]$$

$$= 1.11667\ (correct\ upto\ 5\ decimal\ place)$$

EXAMPLE 2:

B. Composite Simpson's 1/3 Rule: $I = \frac{h}{3}[y_0 + \left(4\sum_{i=1,3,5}^{n-1} f_i\right) + \left(2\sum_{j=2,4,6}^{n-2} f_j\right) + f_n]$

x	1	1.5	2	2.5	3
$f(x) = \frac{1}{x}$	1	0.6666	0.5	0.4	0.3333
	y_0	y_1	y_2	y_3	y_4

$$\therefore I = \frac{h}{3}[y_0 + 4(y_1 + y_3) + 2y_2 + y_4]$$

$$= \frac{1}{6}[1 + 4(0.6666 + 0.4) + 2 * 0.5 + 0.3333]$$

$$= 1.09995(correct\ upto\ 5\ decimal\ place)$$

EXAMPLE 2:

C. Composite Simpson's 3/8 Rule:

$$I = \frac{3h}{8}\left[y_0 + 3\sum_{i=1,4}^{n-1} y_i + 3\sum_{i=2,5}^{n-2} y_i + 2\sum_{j=3}^{n-3} y_j + y_n\right]$$

x	1	1.5	2	2.5	3
$f(x) = \frac{1}{x}$	1	0.6666	0.5	0.4	0.3333
	y_0	y_1	y_2	y_3	y_4

$$\therefore I = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + 3y_3 + y_4]$$

$$= \frac{3}{12}[1 + 3(0.6666 + 0.5) + 3 * 0.4 + 0.3333]$$

$$= \frac{1}{6}(5.6331)$$

$$= 1.00552(correct\ upto\ 5\ decimal\ place)$$

EXAMPLE 3:

Estimate the integral $\int_1^3 \frac{dx}{x}$ using for n=4 and n=6:

A. Composite Trapezoidal Rule

B. Composite Simpson's 1/3 Rule

C. Composite Simpson's 3/8 Rule

Exact Value of $\int_1^3 \frac{dx}{x} = 1.09861$

Solution for n=6:

Here,

$a = 1$

$b = 3$

n=6

$h = (b - a)/n = (3-1)/6 = 2/6 = 1/3 = 0.33333$

EXAMPLE 3:

A. Composite Trapezoidal Rule: $I = \frac{h}{2}[y_0 + \left(2\sum_{i=1}^{n-1} f_i\right) + f_n]$

x	1	1.3333	1.6666	2	2.3333	2.6666	3
$f(x) = \frac{1}{x}$	1	0.750019	0.60002	0.5	0.42858	0.375009	0.3333
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\therefore I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6]$$

$$= \frac{1}{6}[1 + 2(0.750019 + 0.60002 + 0.5 + 0.42858 + 0.375009) + 0.3333]$$

$$= \frac{1}{6} * 6.640556$$

$$= 1.10675(correct\ upto\ 5\ decimal\ place)$$

EXAMPLE 3:

B. Composite Simpson's 1/3 Rule: $I = \frac{h}{3}[y_0 + \left(4\sum_{i=1,3,5}^{n-1} f_i\right) + \left(2\sum_{i=2,4,6}^{n-2} f_i\right) + f_n]$

x	1	1.3333	1.6666	2	2.3333	2.6666	3
$f(x) = \frac{1}{x}$	1	0.750019	0.60002	0.5	0.42858	0.375009	0.3333
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\therefore I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$$

$$= \frac{1}{9}[1 + 4(0.750019 + 0.5 + 0.375009) + 2(0.60002 + 0.42858) + 0.3333]$$

$$= \frac{1}{9}[1 + 4(1.625028) + 2(1.0286) + 0.3333]$$

$$= \frac{1}{9} * 9.890612$$

$$= 1.098956(correct\ upto\ 5\ decimal\ place)$$

EXAMPLE 2:

C. Composite Simpson's 3/8 Rule:

$$I = \frac{3h}{8}\ [y_0 + 3\ \sum_{i=1,4}^{n-1} y_i\ +3\ \sum_{i=2,5}^{n-2} y_i + 2\sum_{j=3}^{n-3} y_j + y_n]$$

x	1	1.3333	1.6666	2	2.3333	2.6666	3
$f(x) = \frac{1}{x}$	1	0.750019	0.60002	0.5	0.42858	0.375009	0.3333
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\therefore I = \frac{3h}{8}[y_0 + 3(y_1 + y_5) + 3(y_2 + y_4) + 2(y_3) + y_6]$$

$$= \frac{1}{8}[1 + 3(0.750019\ + 0.375009\ ) + 3(0.60002\ + 0.42858\ ) + 2 * 0.5 + 0.3333]$$

$$= \frac{1}{8}(8.794214)$$

$$= 1.09928(correct\ upto\ 5\ decimal\ place)$$

IMPORTANT NOTE

Simpson's 1/3 rule is usually the method of preference because it attains third order accuracy with three points, on the other hand Simpson's 3/8 rule required four points.

Simpson's 3/8 rule work better if number of interval is multiple of 3.

Trapezoidal Rule has large truncation error.

We can use Simpson's 3/8 in conjunction with Simpson's 1/3 rule to handle odd number of intervals.

EXAMPLE 4

Numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to b=0.8 using Simpson's 3/8 rule in conjunction with Simpson's 1/3 rule for n=5.

Solution:

The data needed for a five segment application (h=0.16) is

x	0	0.16	0.32	0.48	0.64	0.8
$f(x)$	0.2	1.296919	1.743393	3.186015	3.181929	0.232
	y_0	y_1	y_2	y_3	y_4	y_5

The integral for 1st two segments is obtained using Simpson's 1/3 rule:

$$I = 0.16\left(\frac{0.2+4(1.296919)+1.743393}{3}\right) = 0.3803237$$

The integral for last three segments is obtained using Simpson's 3/8 rule:

$$I = 0.16 * 3\left(\frac{1.743393+3(3.186015\ +3.181929)+0.232}{8}\right) = 1.264754$$

So the total integral is calculated by summing the two result.

$$I = 0.3803237 + 1.264754 = 1.645077$$

EXAMPLE 5

The table bellow shows the temperature $f(t)$ as a function of time.

t	1	2	3	4	5	6	7
$Temperat$ $f(x)$	81	75	80	83	78	70	60
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

A. Use Simpson's $1/3$ method to estimate

$$\int_1^7 f(t)dt$$

B. Use the result in (A) to estimate the average temperature.

EXAMPLE 5:

A. Composite Simpson's 1/3 Rule: $I = \frac{h}{3}[y_0 + \left(4\sum_{i=1,3,5}^{n-1} f_i\right) + \left(2\sum_{i=2,4,6}^{n-2} f_i\right) + f_n]$

t	1	2	3	4	5	6	7
$Temperat$ $f(x)$	81	75	80	83	78	70	60
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$a = 1\,; b = 7; h = \frac{b-a}{6} = \frac{7-1}{6} = 1$

$\therefore I = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$

$= \frac{1}{3}[81 + 4(75 + 83 + 70) + 2(80 + 78) + 60]$

$= 456.33333(correct\ upto\ 5\ decimal\ place)$

B. Average Temperature = $T_{ave} = \frac{\int_1^7 f(t)dt}{b-a} = \frac{456.33333}{6} = 76.055555$

EXAMPLE 6

The velocity of particle is governed by the law:

$$v(t) = \frac{\sin t}{(t+1)^2 \exp(t)}$$

If the initial position of the particle is $x(0) = 0$, then estimate the position $x(2)$ using the integral:

$x(t) = \int_0^t v(t)dt$ by applying suitable newton cotes formula.

EXAMPLE 6

The velocity of particle is governed by the law:

$$f(t) = v(t) = \frac{\sin t}{(t+1)^2 \exp(t)}$$

Given,

$a = 0; b = 2$

Composite Simpson's 1/3 Rule: $I = \frac{h}{3}\left[y_0 + \left(4\sum_{i=1,3,5}^{n-1} f_i\right) + \left(2\sum_{i=2,4,6}^{n-2} f_i\right) + f_n\right]$

$\therefore h = \frac{b-a}{2}$=1

Data for the formula

t	0	1	2
$v(t) = \frac{\sin t}{(t+1)^2 \exp(t)}$	0	0.07739	0.01538

EXAMPLE 6

$I = \frac{h}{3}\ [f(a) + 4f(x1) + f(b)]$

$= \frac{1}{3}[f(0) + 4f(1) + f(2)]]$

$=\frac{1}{3}[(0) + 4 * (0.07739\ ) + (0.01538\ )]\ = 0.108313$

position $x(2) = 0.108313$

EXAMPLE 7

Example: Using Romberg's method compute $I = \int_0^1 e^{-x^2} dx$ correct to 4 decimal places.

Solution: Here

$f(x) = e^{-x^2}$

n=1, 2 and 4

We can take $h = 1, 0.5, 0.25$

i.e,

$h = 1, \frac{h}{2} = 0.5, \frac{h}{4} = 0.25$

EXAMPLE 7

Data for Calculation

x	0	0.25	0.50	0.75	1
$f(x)$	1	0.93941	0.7788008	0.56978	0.36788

Using Trapezoidal Rule with $h = 1$

$$I(h) = I(1) = \frac{1}{2}[(1 + 0.36788)] = 0.68394$$

$$I\left(\frac{h}{2}\right) = I\left(\frac{1}{2}\right) = I(0.5) = \frac{0.5}{2}[(1 + 0.36788) + 2(0.7788008)] = 0.7313704$$

$$I\left(\frac{h}{4}\right) = I\left(\frac{1}{4}\right) = I(0.25) = \frac{0.25}{2}\left[\left((1 + 0.36788) + 2(0.93941 + 0.7788008 + 0.56978)\right)\right]$$

$$= 0.7429827$$

EXAMPLE 7

Now,

$I\left(h, \frac{h}{2}\right) = I(1, 0.5) = \frac{1}{3}[4 * I(0.5) - I(1)] = \frac{1}{3}[4 * 0.7313704 - 0.68394] = 0.74718053$

$I\left(\frac{h}{2}, \frac{h}{4}\right) = I(0.5, 0.25) = \frac{1}{3}[4 * I(0.25) - I(0.)] = \frac{1}{3}[4 * 0.7429827 - 0.7313704] = 0.74685347$

$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = I(1, 0.5, 0.25) = \frac{1}{3}[4 * I(0.5, 0.25) - I(1, 0.5)] = \frac{1}{3}[4 * 0.74685347 - 0.74718053] = 0.746744$

EXAMPLE 7

0.68394		
0.7313704	0.74718053	
0.7429827	0.74685347	0.74674445

$$\therefore \boldsymbol{I} = \int_{\boldsymbol{0}}^{\boldsymbol{1}} \boldsymbol{e}^{-\boldsymbol{x}^{2}} \boldsymbol{dx} = 0.74674445$$

Absolute Error = True value − approximate value

$$= 0.74682 - 0.74674445 = 0.00007555 = O(h^4)$$