Numerical Differentiation

Raqeebir Rab

Example of Numerical differentiation

- Problem statement:

Estimate the first order derivative of $f(x) = lnx$ at x = 1. Using the following formula:

1. First order forward difference formula
2. First order backward difference formula
3. First order central difference formula.

Compare the results with exact value 1. Assume step size h = 0.1

Example of Numerical differentiation

1.1st order Two Points Forward Difference Formula

$$f(x) = lnx$$

x	$f(x)$
1	0
1.1	0.0953101

Relative % Error

$$\epsilon = \frac{|1 - 0.953101\ |}{1} \times 100\%$$

$$= 4.69\%$$

Two Points Forward Difference Formula

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$f'(1) = \frac{f(1+0.1) - f(1)}{0.1}$$

$$= \frac{ln(1.1) - ln(1)}{0.1}$$

$$= \frac{0.0953101 - 0}{0.1}$$

$$= 0.953101$$

Example of Numerical differentiation

3.2nd order Three Points Central Difference Formula

$$f(x) = lnx$$

x	$f(x)$
0.9	-0.10536052
1	0
1.1	0.0953101

Relative % Error

$$\epsilon = \frac{|1 - 1.003531|}{1} \times 100\%$$
$$= 0.35\%$$

Three Points Central Difference Formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(1) = \frac{f(1.1) - f(0.9)}{2(0.1)}$$

$$= \frac{ln(1.1) - ln(0.9)}{0.2}$$

$$= \frac{0.0953101 - (-0.10536052)}{0.2}$$

$$= 1.003531$$

Example of Numerical differentiation

2. 1st order Two Points Backward Difference Formula

$$f(x) = lnx$$

x	$f(x)$
0.9	-0.10536052
1	0

Relative % Error

$$\epsilon = \frac{|1 - 1.0536052\ |}{1} \times 100\%$$

$$= 5.36\%$$

Two Points Backward Difference Formula

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$f'(1) = \frac{f(1) - f(1-0.1)}{0.1}$$

$$= \frac{ln(1) - ln(0.9)}{0.1}$$

$$= \frac{0 - (-0.10536052)}{0.1}$$

$$= 1.0536052$$

Example of Higher Order Numerical differentiation

- Problem statement:

Use three points formula to estimate the 2nd order derivative of $f(x) = \frac{1}{1+x^2}$ at *x* = 0.5, with step size *h* = 0.01. Compare the result with true value.

Solution:

$$f(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$$

$$= \frac{2(3(0.5)^2-1)}{((0.5)^2+1)^3}$$

$$= \frac{-0.5}{1.953125}$$

$$= -0.256$$

Example of Higher Order Numerical differentiation

2nd order Three Points Difference Formula

$$f(x) = \frac{1}{1+x^2}$$

x	$f(x)$
0.49	0.80638658172728005
0.5	0.8
0.51	0.7935878104912308

Relative % Error

$$\in = \frac{|-0.256 \ + 0.256077|}{-0.256} \times 100\%$$
$$= 0.03\%$$

2nd Order Three Points Difference Formula

$$f''(x) = \frac{f(x+h) - 2f(x) + \ f(x-h)}{h^2}$$

$$f''(0.5) = \frac{f(0.51) - 2 * (0.5) + f(0.49)}{(0.01)^2}$$

$$= \frac{0.7935878104912308 \ - 2 * (0.8) + 0.80638658172728005}{0.0001}$$

$$= \frac{1.59997439221851085 - 1.6}{0.0001}$$

$$= \frac{-0.000025607}{0.0001} = -0.256077$$

Example of Numerical differentiation

- Current through a capacitor is given by $I(t) = c\frac{dv}{dt} = cv'(t)$ where $v(t)$ is the voltage accross the capacitor at time t and C is the capacitance value of the capacitor. Estimate the current through the capacitor at t=0.5 using central difference formula. Assume the following:

$$v(t) = (t + 0.1)e^{\sqrt{t}}$$
$$C = 2F$$

Example of Numerical differentiation

$f(t-h) = f(0.5-0.2) = v(0.3) = (0.3+0.1)e^{\sqrt{0.3}} = 0.69172405014$
$f(t) = f(0.5) = v(0.5) = (0.5+0.1)e^{\sqrt{0.5}} = 1.21686898899$
$f(t+h) = f(0.5+0.2) = v(0.7) = (0.7+0.1)e^{\sqrt{0.7}} = 1.84691462256$

x	$f(x)$ = v(t)
0.3	0.69172405014
0.5	1.21686898899
0.7	1.84691462256

$I(t) = c\frac{dv}{dt} = cv'(t)$ = 2*(2.88797643104)
=5.77595amp

Three Points Central Difference Formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(0.5) = \frac{f(0.7) - f(0.3)}{2(0.2)}$$

$$= \frac{1.84691462256 - 0.69172405014}{0.2}$$

$= 2.88797643104$

Example of Numerical differentiation

- Problem statement:

A jet fighter's position on an aircraft carrier's runway was timed during landing:

t(s)	0	2	4	6	8	10	12	14	16
x (m)	0	0.7	1.8	3.4	5.1	6.3	7.3	8.0	8.4

Where x is the distance from the end of the carrier. Estimate:

(a) Velocity ($\frac{dx}{dt}$)and

(b) Acceleration ($\frac{dv}{dt}$)

At t = 10 using numerical differentiation.

Example of Higher Order Numerical differentiation

Velocity at $t = 10, f'(t) = v(t) = \frac{dx}{dt}$

t	$f(t)$ = $x(t)$
10	6.3
12	7.3
14	8.0

So, velocity at t=10, $v(t)$ = 0.575 m/s

Three-point forward difference formula

$$f'(t) = \frac{-3x(t) + 4x(t+h) - x(t+2h)}{2h}$$

Here h=2

$$f'(10) = \frac{-3(6.3) + 4(7.3) - (8.0)}{2 * 2}$$

$$= \frac{-18.9 + 29.2 - 8.0}{4}$$

$$= \frac{2.3}{4} = 0.575$$

Example of Higher Order Numerical differentiation

Velocity at $\mathrm{t} = 10, f'(\mathrm{t}) = \mathrm{v}(\mathrm{t}) = \frac{dx}{dt}$

t	$f(t)$ = $x(t)$
10	6.3
12	7.3
14	8.0

So, velocity at t=10,
$v(t)$ = 0.575 m/s

Three-point forward difference formula

$$f'(t) = \frac{-3x(t) + 4x(t+h) - x(t+2h)}{2h}$$

Here h=2

$$f'(10) = \frac{-3(6.3\ ) + 4(7.3\ ) - (8.0\ )}{2 * 2}$$

$$= \frac{-18.9 + 29.2 - 8.0}{4}$$

$$= \frac{2.3}{4} = 0.575$$

Example of Higher Order Numerical differentiation

Velocity at $t = 10, f'(t) = v(t) = \frac{dx}{dt}$

t	$f(t) = x(t)$
8	5.1
10	6.3
12	7.3

So, velocity at t=10,
$v(t)$ = 0.55 m/s

Three-point central difference formula

$$f'(t) = \frac{x(t+h) - x(t-h)}{2h}$$

Here h=2

$$f'(10) = \frac{7.3 - 5.1}{2 * 2}$$

$$= \frac{2.2}{4} = 0.55$$

Example of Higher Order Numerical differentiation

Acceleration at $t = 10, f''(t) = v'(t) = a(t) = \frac{dv}{dt} = \frac{d^2x}{d^2t}$

t	$f(t)$ = $x(t)$
8	5.1
10	6.3
12	7.3

So, acceleration at t=10, $a(t)$ = 0.05 m/s^2

2nd order three-point central difference formula

$$f''(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}$$

Here h=2

$$f''(10) = \frac{7.3 - 2 * 6.3 + 5.1}{4}$$

$$= \frac{0.2}{4} = 0.05$$

Example of Numerical Differentiation

- Problem statement: Compute from the following table the value of derivative of $y = f(x)$ at x=1.7489 using Newton Forward interpolating formula.

x	$f(x)$
1.73	1.77284
1.74	1.55204
1.75	1.73773
1.76	1.72044
1.77	1.703329

Example of Numerical Differentiation

- Newton Forward Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.73	1.77284				
		-0.2208			
1.74	1.55204		0.40649		
		0.18569		-0.60947	
1.75	1.73773		-0.20298		0.812629
		-0.01729		0.203159	
1.76	1.72044		0.000179		
		-0.017111			
1.77	1.703329				

Example of Numerical Differentiation

- The value of *x* at you want to find $f(x) at$ x= 1.7489

$h = x_1 - x_0 = 1.74 - 1.73 = 0.01$

$u = \frac{x - x_0}{h} = \frac{1.7489 - 1.73}{0.01}$ = 1.89

Newton forward interpolation formula:

y= $y_0 + u\Delta y_0 + u(u-1)\frac{\Delta y_0}{2!} + u(u-1)(u-2)\frac{\Delta^2 y_0}{3!} + u(u-1)(u-2)(u-3)\frac{\Delta^3 y_0}{4!} + u(u-1)(u-2)(u-3)(u-4)\frac{\Delta^4 y_0}{5!}$

Now differentiating y with respect to x:

$\frac{dy}{dx} = \frac{1}{h}(\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0)$

$= \frac{1}{0.01}(-0.2208 + \frac{2*1.89-1}{2} 0.40649 + \frac{3(1.89)^2-6*1.89+2}{6} -0.60949 + \frac{4(1.89)^3-18(1.89)^2+22*1.89-6}{24} 0.812629)$

$= \frac{1}{0.01}(-0.2208 + \frac{2.782}{2} 0.40649 + \frac{1.3763}{6} -0.60949 + \frac{-1.712724}{4!} 0.812629)$

1st derivative of $y = f(x)\ at\ x = 1.7489$

$\frac{dy}{dx} = 14.642679$

Example of Numerical Differentiation

- Problem statement:

Given below the table of function values of $f(x) = \sinh(x)$. Estimate the first derivative at x = 1.3 with h = 0.1, using three-point centrel difference formula. Compute an improved estimate using Richardson Extrapolation. Exact value of $f'(x) = \cosh(1.3) = 1.9709$.

x	1.1	1.2	1.3	1.4	1.5
$f(x)$	1.3356	1.5095	1.6983	1.9043	2.1293

Example of Numerical differentiation

t	$f(t) = x(t)$
1.2	1.5095
1.3	1.6983
1.4	1.9043

$f'(x)$= 1.974

Three-point central difference formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Here h=.1

$$f'(10) = \frac{1.9043 \; - 1.5095}{2 * .1}$$

=1.974

Example of Numerical differentiation

Richardson extrapolation

t	$f(t)$ = $x(t)$
1.2	1.5095
1.3	1.6983
1.4	1.9043

$f'(x)$= 1.9745

Three-point central difference formula

$$D(h) = \frac{f(x+h) - f(x-h)}{2h}$$

Here h=.1

$$D(h) = \frac{1.9043 \ - 1.5095}{2 * .1}$$

=1.9745

Example of Numerical differentiation

Richardson extrapolation

t	$f(t) = x(t)$
1.25	1.60191
1.3	1.6983
1.35	1.79909

$f'(x)$= 1. 9718

Three-point central difference formula

$$D(rh) = \frac{f(x+rh) - f(x-rh)}{2rh}$$

Here h=.1 r=.5
rh = 0.05

$$D(rh) = \frac{1.79909 - 1.60191}{0.1}$$

$=$1.9718

Example of Numerical differentiation

Richardson extrapolation

$f'(x)$= 1.9709

Three-point central difference formula

$$f'(x) = \frac{D(rh) - r^2 D(h)}{1 - r^2}$$

$$f'(x) = \frac{1.9718 - (0.25)1.945}{1 - 0.25}$$

=1.9709